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ANALYTIC MODELING OF SEVERE VORTICAL STORMS

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20. Abstract

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   Since the one-cell storm is believed to be largely described already for
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In particular, the properties and location of the eye wall are sought, since
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   The modeling proceeds from basic thermohydrodynamic principles. Non-
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details of particular parameterizations, is being avoided as much as possible.

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PREFACE

The participants in this program are Francis Fendell, principal investigator, and Phillip Feldman, numerical analyst, of TRW Defense and Space Systems Group, and George Carrier of Harvard University, Cambridge, MA, consultant to TRW. Also contributing was Paul Dergarabedian of the senior staff of the Energy Systems Division of TRW, now retired.

The participants are grateful to project technical monitors Walter Martin and James Hughes of the Atmospheric Sciences Program of the Office of Naval Research for their encouragement and their patience.

SUMMARY

Analytic modeling of well-organized rotating convective storms is being taken to elucidate the evolution from a moderately intense one-cell vortex, characterized by low-level pressure deficits on the order of one percent of atmospheric pressure, to a very intense two-cell vortex, characterized by low-level pressure deficits on the order of ten percent of atmospheric pressure. The physical distinction between the two stages is the insertion of a dry, compressionally heated, nonrotating, central downdraft of originally tropopause-level air in the more severe case. The quasi-steady mature description of the thermohydrodynamic structure of each vortex is being developed, and then the conditions for transition from the moderately intense to the very intense vortex are to be sought. The practical motivation is to make progress toward the highly desirable, but very formidable task of being able to anticipate which tropical storms or minimal hurricanes will evolve to supertyphoons.

Since the one-cell storm is believed to be largely described already for present purposes, description of the two-cell vortex is the current challenge. In particular, the properties and location of the eye wall are sought, since the potential-vortex and surface-inflow subdivisions of the structure of a very intense vortex are in hand.

The modeling proceeds from basic thermohydrodynamic principles. Nonessential geometric detail, as well as association of conclusions with the details of particular parameterizations, is being avoided as much as possible.

I. INTRODUCTION

(a) STATEMENT OF THE OBJECTIVE

The principal contribution still to be made in forecasting intensity of tropical storms is to identify the observable(s) that permit one to anticipate (as much in advance as possible) which incipient storms will

become hurricanes and which will not.[#]

Very roughly about half of all tropical storms, one-cell vortices extending vertically from sea level to torpopause and radially many hundreds of miles, transform into two-cell vortices (hurricanes) (Fendell 1974). The transition involves insertion of a column of relatively dry, clear, nonrotating, originally tropopause-level, compressionally heated, slowly recirculated air at the center of the vortex. Accordingly, the region of intensely swirling, very cloudy, torrentially raining updraft becomes displaced to an annulus. An "eye" has then been inserted within an "eyewall". A formidable vortex, with peak swirl of perhaps not 100 mph and horizontal pressure deficit at sea level of a few tens of millibars (the tropical storm, as defined here), has transformed into a terrifying vortex of possibly over 200 mph peak swirl and a pressure deficit of 100 mb or more (the hurricane, as defined here). The question arises, what telltale observable accessible to aircraft, satellites, etc., would permit one to anticipate the transition from tropical storm to hurricane? Of course, the intensity of a tropical cyclone is not a simple monotonic increase with time to a peak, following by a simple monotonic decay; thus, there may be several transitions back and forth between one- and two-cell structure, suggestive of partial insertion, then removal, of an "eye" (Carrier 1971a).

The initial goal of this project is the formidable one of delineating the structure of a well-developed two-cell vortex, to complement the relatively well-understood structure of a one-cell vortex. Before subleties of the transition can be detected, the "stable equilibrium states" should be characterized in sufficient quantitative detail.

Normally quantifying any details about structure in tropical cyclones

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[#] Gray (1979, p. 51) comments: "Tropical cyclone forecasters generally agree that skill at operational forecasting of tropical intensity change is nearly zero." He notes that trying to relate intensity to cloudiness configuration (as observed by passive satellites) is subject to significantly erroneous results, and furnishes inadequate detail. It may be noted that the gustiness of winds is not discussed, since treatment of the variance seems premature when prediction of the mean is still not possible. Gustiness factors for tropical cyclones (ratio of peak transient wind to wind sustained over a scale of one minute to five minutes, depending on the correlator) range from 1.0 to 1.8 in the literature (Brand et al 1979).

is highly uncertain because proper formulation of turbulent diffusion, cummulus convection, and radiative transfer is unknown. These three important small-scale phenomena may well not have satisfactory representation in terms of macroscale variables, so even the very attempt at such parameterization is doomed at the outset. Yet such parameterization is the essence of all contemporary tropical cyclone modeling, and any new undertaking could just augment the plethora of almost totally uncorroborated speculation. Why is there cause for optimism?

(b) AN APPROACH TO TWO-CELL STRUCTURE FOR AN INTENSE ATMOSPHERE VORTEX

The answer concerns a little ingenuity, a little insight, and a lot of luck.

The ingenuity lies in the fact that, whereas almost all contemporary hurricane modeling entails direct numerical assault on all-inclusive mathematical formulation, this project utilizes an alternative, approximate analytic treatment as much as possible. While other liabilities may be cited (e.g., finite-differencing introduces spurious effects not present in the physical formulation), there is one particular difficulty with direct numerical computation of special relevance here: a physical effect important anywhere in the flow field must be retained everywhere in the flow field. Thus rather arbitrary models for turbulent diffusion, cumulus convection, and radiative transfer are present, possibly to vitate results throughout the flow field. In the alternative approximate aralytic approach examined here (Carrier 1970; Carrier, Hammond and George 1971), the hurricane is subdivided logically into those parts in which different physical phenomena dominate; when a particular subdivision is treated, only that subset of the full equations that retains the locally dominant physical processes is solved. Such a subset is usually more tractable to solve, and parametric variation easier to examine. A composite global solution is synthesized from solutions to the subdomains by demanding appropriate continuity of dependent variables and their fluxes at interfaces.

The little insight referred to above is the deduction that, under the high-speed portion (only) of a severe rotating storm, the nonlinear "surface" inflow layer is inviscidly controlled; only in a very small sublayer (immediately continguous to the ground) of the inflow layer does diffusion

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enforce the no-slip boundary condition (Carrier 1971b; Burggraf, Stewartson and Belcher 1971; Carrier and Fendell 1978). The diffusive sublayer becomes thinner, and the outer inflow layer thicker, as one moves in toward the center of the vortex. Why this is so is subtle, but it is familiar in nonrotating contexts to aerodynamicists in the form of the following bit of well-known empiricism (e.g., Launder 1964): a near-wall diffusive layer becomes thinner in the direction of a favorable (i.e., accelerating) pressure gradient. At sea level, the pressure is high at the lateral edge of an intense vortex and the pressure is greatly reduced at the center, such that a strong positive pressure gradients exists; thus, the low-level swirling influx (that erupts up the eyewall) is predominantly inviscidly controlled as it proceeds toward the hurricane center.

Now, radiative cooling is important only in the outer and less cloudy portions of the storm (Fendell 1974), and the simple well-defined model of moist-adiabatic ascent* suffices to describe the locus of thermodynamic states in the eyewall. In the previous paragraph it is pointed out that the low-level flow entering the eyewall is nondiffusive. This is the crucial ingredient of good fortune that joins ingenuity and insight. Two of the three small-scale processes that are the bane of the numerical modeler (specifically, turbulent diffusion and radiative transfer) do not enter significantly in the portion of the hurricane flow field critical to quantitative analysis of two-cell structure, and the third process (cumulus convection) can be formulated locally in a well-accepted, well-defined form.

The upshot is the following good fortune: one of the few questions about hurricane structure that can be definitively formulated by contemporary modeling happens to be one of the most important. Specifically, that question is, what is the balance of forces and thermodynamic states in the central region of a tropical storm that permits two-cell structure to arise as an alternative to one-cell structure?

* Water-vapor-containing air rises, expands and cools dry-adiabatically until saturation. Thenceforth, it continues to ascend such that saturation of water vapor is maintained at the local thermodynamic state, the condensed excess water substance falling out so that it is unavailable for re-evaporative cooling. (Hence the process cannot be reversed.) The originally low-level, convectively unstable fluid undergoing this ascent is taken to rise too quickly to entrain any ambient air, and hence remains unmixed with cooler or drier air. In fact, of course, some mixing with nearby ambient air occurs and some re-evaporation into this unsaturated mix also occurs, but the no-mixing case is realistic enough to provide valuable insight.

(c) THE BOUNDARY-VALUE PROBLEM

For axisymmetric inviscid swirling flow in which a radial influx becomes an axial upflux, the boundary-value problem is well-defined. Especially if one confines attention at the outset to low altitudes such that incompressibility suffices, the formulation is available in texts (e.g., Batchelor 1967). Angular momentum is constant on streamlines, continuity is enforced, advective and centripetal accelerations balance the radial pressure gradient, and the total pressure head is constant across streamlines (Carrier, Dergarabedian and Fendell 1969). The annulus is bounded on one side by a free streamsurface that demarcates the eyewall interface with the eye, and on the other side by a free streamsurface that demarcates the eyewall interface with the outer vortex. The position of both these bounding streamsurfaces must be found in the course of solution, under enforcement of statements concerning the continuity of pressure. One result sought is the displacement of the eyewall from the axis of symmetry, which is also the axis of rotation. It is anticipated that this displacement ultimately increases with height.[#] The mathematically

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[#] The increasing displacement of the eyewall from the axis with altitude seems likely under conservation of angular momentum, since density decreases with altitude. Conservation of mass also suggests that the cross-sectional area of the eyewall annulus increases with altitude. The fact that the eyewall tilts radially outward with increasing height is suggested further by certain satellite photographs of hurricanes (Fendell 1974). This outward-sloping behavior may seem just another passing observation, but actually it bears directly on perhaps the most controversial question dividing hurricane models during the last three decades. That question concerns the amount of augmented sensible and latent heat transfer (above ambient tropical autumnal transfer rates) from ocean to atmosphere that is necessary near the center of a hurricane to sustain the hurricane (Carrier, Hammond and George 1971). If the eyewall were vertical, augmentation of enthalpy transfer from sea to air would have to be an order of magnitude greater than conventional rates (Malkus and Riehl 1960); pressure deficits achievable under compressional heating in the eye could not serve as an explanation of intense swirl speeds observed in the eyewall, and instead oceanic heat transfer would have to explain the density reduction that hydrostatically integrates to low pressure at the base of the eyewall. Such an increase in oceanic transfer seems mechanistically unlikely. If the eyewall slopes radially outward, no such augmentation in enthalpy transfer is requisite, because low pressure at the base of the eyewall could be explained by density reduction through compressional heating in the upper troposphere.

elliptic character of the boundary-value problem reflects the physical fact that streamwise and transverse derivatives, and streamwise and transverse velocity components, are of comparable magnitude; there is no single preferred direction of change. Indeed, frankly, a great deal of time has been unsuccessfully expended by the investigators seeking a valid mathematical transformation that would convert the elliptic problem into a parabolic one; an axisymmetric parabolic problem is more readily solved precisely because it possesses a single dominant direction of change. Such a transformation seemed attained, but it ultimately proved a chimera, and now the investigators must come to grips with the nonlinear elliptic problem.

Some preliminary insight is attainable at the sacrifice of detail by using an "integral method" (Finlayson 1972). Roughly, one "forces" a parabolic-like character on the problem by taking transverse averages and then seeking streamwise variance of these averages from an appropriately simplified boundary-value problem. Such gross resolution can suffice as preliminary guidance only, and ultimately greater resolution is required for present purposes.

(d) ON SIMILARITIES OF SWIRL-STABILIZED-COMBUSTOR FLOWS AND HURRICANE DYNAMICS

The meteorologically motivated swirling-flow studies being discussed should be of value in aerodynamically stabilized continuous combustors. Such combustors are used in face-fired furnaces (industrial boilers, whether fueled by oil or coal) and in gas turbines. In fact, the dual-purpose pertinence of the proposed research is such that the matter is now developed at some length.

For fuel efficiency, low pollution, and flame stabilization, combustors with recirculatory zones are quite common (Beer and Chigier 1972). For premixed fuel and air, such a reversed-flow domain lets one flow the fresh charge at speeds in excess of the flame speed, without having blow-off. For diffusion flames (unpremixed fuel and air), one can retain the reactants longer in the combustion zone if either reduced volatility or slow gas-phase reaction rates (relative to flow rates) poses a problem. One may view the mechanism by which the recirculation zone succeeds in bringing continuously

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introduced fresh charge to its ignition condition in several ways. The
recirculatory "bubble" retains heat and hot radicals to raise the charge
temperautre and to instigate chain-branching chemical mechanisms. Also,
the stagnation-point-generating recirculatory "bubble" reduces flow speed
relative to reaction rate, with the result that less incompletely-burned
reactant escapes; such "lost" reactant is not only economically wasteful
and polluting, but energetically unfavorable, because some sensible heat
is convected away without exploitation of the full chemical exothermicity
possible.

Recirculation may be achieved in the wakes of bluff bodies introduced
into the flow, but these entail large pressure losses in high-speed flow,
and in general introduce materials-survival and coke-deposition problems.
In recent years, abrupt area expansion of the flow, secondary air jets,
and other adverse-pressure-gradient-generating designs have been implemented
to achieve recirulatory flow aerodynamically, i.e. without introduction
of physical obstacles into the flow (Syred, Chigier and Beer 1971). However,
of prinicpal interest here, is the use of swirling flow at the inlet, either
of both the fuel and air stream, or at least of the air stream alone, to
aid achievement of recirculatory flow in the reaction zone of the combustor.

In general in swirling-stabilized combustors, the reactants enter the
combustor with axial (i.e., streamwise) momentum and with angular momentum;
the ratio of the latter to the former is defined as the swirl ratio and
traditionally denoted S (Beer and Chigier 1972). [Some authors (e.g.,
Hall 1972) define an equivalent angle of swirl, ϕ, by the definition $\tan \phi \equiv$
(v/w), where v denotes the swirl component of the velocity, and w denotes
the axial component.] Typically, for $S < 0.6$, the swirl causes some
streamline spreading and central pressure reduction, but the rotating is
too weak to induce recirculation. For $S > 0.6$ or so, there may be toroidal
recirculation zones, but the flow in the immediate vicinity of the axis of
rotation may be downwind at all axial positions. For $S > 1$, an "equivalent-
bluff-body", or bubble-type, recirculation zone often lies about the axis
of rotation. This large-swirl case is the one of interest here. In
furnaces, the "bubble" can be open-ended in that the recirculation zone
extends from near the inlet down the entire axial length of the chamber,
with the result being a nearly columnar flow field (Khalil, El-Mahallawy
and Moneib 1976).

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For such large-swirl cases, one has a flow that is well described in time avarage by an inviscid incompressible axisymmetric swirling model (Bossel 1973). In fact, the relevant quasilinear second-order elliptical partial differential equation is precisely that which is there believed pertinent to the two-cell severe atmospheric vortex. The vortex-breakdown analogy between severe-atmospheric-vortex flows and recirculatory combustor flows seems more than a possible superficial resemblance to be noted in passing. The equation is usually expressed in cylindrical polar coordinates, and involves both axial and radial partial derivatives, and is not fully specified until flow conditions pertinent at the inlet and at the outlet are specified (Leibovich 1978).

That the mathematical formulation of the strong-swirl-stabilized furnace flow is so similar to that of the two-cell atmospheric vortex is, in restrospect, obvious, because the physical phenomena have appreciable similarities (Carrier, Fendell and Feldman 1980). In both instances, there is an annular flow displaced from the axis of symmetry, in which rapidly swirling flow moves axially "downwind" (the "eyewall"). Enveloping the axis there is a "filling" flow whose axial direction of movement is partly reversed to that of the annular fluid. This reversed-flow fluid in the "bubble" (or "eye") may be swirling, but the principal interest is in (1) ascertaining the location of the interface distinguishing annular (eyewall) fluid from bubble (eye) fluid, and (2) details of the annular flow outside the bubble. (What flow there is inside the bubble may be intricate and it is probably not critical to ascertain the details for many purposes.)

It is not the objective here to deal exhaustively with similarities (and also distinctions) between recirculatory bubble-type swirling furnace flows and two-cell severe atmospheric vortical storms. Nevertheless, several points may be added. First, the outer boundary on the annular flow is prescribed in the case of a furnace: in general it is the impervious outer solid wall of the inlet-diffuser-dump geometry. For the atmospheric vortex, there is no lateral surface, and a second free bounding surface must be identified in the course of solution: it is the (nonsolid) boundary between the eyewall and outer vortex of the hurricane. Thus, the atmospheric-vortex case has a free surface bounding the eyewall on both its near-axis and outer limits; the atmospheric-vortex case is more difficult in this

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regard. Second, whereas the swirling in the annulus is often of rigid-body type in a combustor, an analytically simple special form that reduces the quasilinear partial differential equation to a far more tractable linear equation, the swirling in the annulus in the atmospheric vortex is likely to be of a more complicated, less tractable nature. Nevertheless, the total pressure head is invariant from one streamline to another in the atmospheric vortex, while it is almost certainly not invariant in the furnace. While these characteristics are somewhat off-setting in the complication that they entail, the atmospheric vortex case ends up the more challenging. Third, in both flows there are heat releases: from chemical exothermicity in the furnace, and from the latent heat of condensation in the hurricane. In neither case is this heat release by itself important enough to necessitate introduction into the dynamics of compressibility for a physically useful quantification. However, the atmospheric vortex extends from ground level to tropopause, and over this scale of altitude compressibility enters; of course, a furnace certainly has no comparable extent. Nevertheless, if one confines attention to just the very-near-ground portion of the eye/eyewall interface in a two-cell vortex (a portion referred to as the turnaround or corner, in that low-level swirling influx becomes swirling upflux), incom- pressibility suffices for the atmospheric vortex as well. Fourth, one of the challenging matters in the elliptic flows that characterize bubble-type furnace flows and atmospheric vortex flows is appropriate characterization of the "outlet" flow. Elliptic flows require specification of flow constraints at all "boundaries"; such specifications at the inlet and the lateral "boundaries" are straightforward to state physically even if sometimes awkward to manipulate mathematically. However, one has less experience concerning what is appropriately specified at the exit plane, such that the formulation is neither underspecified or overconstrained. It is to be remembered that the exit plane of the annular fluid may be the entrance plane of the core ("bubble" or "eye") fluid, and the mathematics must be "informed" about the properites of such mass as enters the computational domain. In the case of a hurricane it is known that the well-developed eye extends from tropopause to virtually sea level and contains virtually nonrotating air, whereas an open-ended reversed-flow domain in a swirling furnace may have appreciable swirl within the air entering at the downwind extremity.

2. A BOUNDARY-VALUE PROBLEM FOR THE TURNAROUND

An inviscid incompressible steady axisymmetric model of the turn-around region in a severe vertical vortex is adopted, in cylindrical polar coordinates with origin at the ground on the axis of rotation (and of symmetry) (Fig. 1).

The azimuthal, radial, and axial velocity components (v^*, u^*, and w^*, respectively), in view of conservation of angular momentum and of conservation of mass, may be written [$\Gamma^* = r_1^* V^*$, where r_1^*, is the (given) radius of peak swirl in region I, and where V^* is the (given) value of that peak swirl]

$$(r^* v^*)^2 = \Gamma^{*2} F(\psi); \tag{1}$$

$$-r^* u^* = \frac{\Gamma^* h^*}{\pi} \frac{\partial \psi}{\partial z^*} ; \tag{2}$$

$$r^* w^* = \frac{\Gamma^* h^*}{\pi} \frac{\partial \psi}{\partial r^*} . \tag{3}$$

Super asterisk denotes a dimensional quantity; no asterisk, a dimensionless quantity. The cylindrical radial coordinate is r^*; the axial coordinate, z^*; the maximum height of the surface inflow layer, h^*; the streamfunction for the secondary flow (involving velocity components u^*, w^*, only), ψ; the function giving distribution of angular momentum with streamfunction, F.

Conservation of radial momentum is (ρ_d^* is density)

$$u^* \frac{\partial u^*}{\partial r^*} + w^* \frac{\partial u^*}{\partial z^*} + \frac{\partial}{\partial r^*} \left(\frac{p^*}{\rho_d^*} \right) = \frac{\Gamma^{*2} F(\psi)}{r^{*3}} . \tag{4}$$

Bernoulli's equation is

$$\frac{u^{*2} + w^{*2}}{2} + \frac{p^*}{\rho_d^*} + \frac{\Gamma^{*2} F(\psi)}{2 r^{*2}} + g^* z^* = \frac{\Gamma^{*2}}{2 r_1^{*2}} + \frac{p_1^*}{\rho_d^*} ; \tag{5}$$

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in (5) the Bernoulli constant is taken as universal for all streamlines emerging from region II in view of the known relation

$$\frac{[u^*(r_1^*,z^*)]^2}{2} + \frac{\Gamma^{*2}F(\psi)}{2r_1^{*2}} = \frac{\Gamma^{*2}}{2r_1^{*2}} \qquad (6)$$

and the fact that the pressure above hydrostatic is axially invariant for $z^* \leq h^*$, within the conventional boundary-layer approximation. Within the cyclostrophic approximation,

$$p_a^* - p_1^* \doteq (\rho_d^*/2)V^{*2} = (\rho_d^*/2)(\Gamma^{*2}/r_1^{*2}), \qquad (7)$$

where p_a^* is the ground-level __ambient__ pressure (given).

From a crude treatment of the turnaround, to be published elsewhere, $(h^*/r_1^*) \doteq 0.2$ for relevant values of r_1^*, V^*, where typical (given) values, for a severe tornado, are $r_1^* \doteq 160$ m, $V^* \doteq 100$ m/s. In any case, h^* and r_1^* are taken as known. Thus, in addition to its use in (2) and (3), h^* is employed to nondimensionalize the independent variables:

$$x = \pi r^*/h^* \quad , \quad y = \pi z^*/h^*. \qquad (8)$$

Subtraction of the radial derivative of (5) from (4) gives

$$x\left(\frac{1}{x}\psi_x\right)_x + \psi_{yy} + \frac{F'(\psi)}{2} = 0 , \qquad (9)$$

where subscripts x and y denote partial differentiation. The swirl profile $F(\psi)$ is known from conditions holding at $x = x_1 = \pi r_1^*/h^* \equiv \pi/h$. It is known from published solution of region II that F is a monotonically increasing function of ψ, such that $F(0) = 0$ and $F(I) = 1$, where the datum $\psi(x>x_1,0) = 0$ is adopted,

and from (2) and (8),

$$\psi(x_1, \pi) = \int_0^\pi u(x_1, \zeta) \, d\zeta \equiv I , \qquad (10)$$

where $u = u^*/V^*$, and, for completeness, $v = v^*/V^*$. Clearly, for F linear in ψ, (9) becomes Poisson's equation, and for F quadratic in ψ, (9) becomes Helmholtz's equation. Although, from known results for region II, such simple forms for $F(\psi)$ are not detailed replications for $(r^*v^*)/\Gamma^*$ of interest, still their tractability urges their adoption.

The pressure along the streamline that separates from $y = 0$ at $x = x_1$ remains at p_1^*, for the model of a nonrotating "eye" isobaric at altitudes at which gravity plays no role. Then, from (5), since $F(0) = 0$,

$$(\psi_x)^2 + (\psi_y)^2 = \left(\frac{x}{x_1}\right)^2 \quad \text{on } \psi = 0 . \qquad (11)$$

The pressure along the streamsurface $\psi = I$, which passes through the circle $x = x_1$, $y = \pi$, should be consistent with the pressure in the potential-vortex region; hence

$$p(x,y) = p_1 + \left[1 - \left(\frac{x_1}{x}\right)^2\right] , \qquad (12)$$

from the cyclostrophic balance, where the nondimensionalization

$$p = \frac{p^*}{\frac{1}{2}\rho_d^* V^{*2}} \qquad (13)$$

has been adopted. From (5), (12), and (13), with neglect of the gravitational term, since $F(I) = 1$,

$$\psi_x^2 + \psi_y^2 = 0 \quad \text{on } \psi = I . \qquad (14)$$

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This condition merely states that there is no contribution from the secondary-flow velocity components u*, w* to the pressure field, where region III interfaces with region I.

Equations (9), (11), and (14) constitute the boundary-value problem of interest, when supplemented by a relation for $F(\psi)$, obtained from previously executed analysis of region II. By (1), (2), (6), and the fact that $w(x_1,y) \doteq 0$, the relation for $F(\psi)$ gives implicitly an expression for $\psi(x_1,y)$, $0 \leq y \leq \pi$, with which to initiate the analysis (see below). Finally, it is anticipated that $\psi(x,y)$, as given by the boundary-value problem is periodic in y; this periodicity reflects the fact that the formulation must be revised after completion of about one cycle. More explicitly, it is anticipated that the locus of any streamline, say $\psi = I/2$, as y increases, decreases to a minimum value of x, then increases in x to recover its initial value $x = x_1$ and to achieve a peak value in x, before decreasing back to $x = x_1$ to begin another period. It is to be expected that the undulating flow so generated will be unstable somewhere along its trajectory, but our present job is to find that flow and the stability problem is deferred until that is done. Of particular interest are the amount by which v exceeds the value associated with $x = x_1$, $\psi = I$, and the values of x and ψ at which the maximum occurs.

For explicitness, a _rough_ characterization of the function $\psi(x_1,y)$, $0 \leq y \leq \pi$, is now presented. In the largely inviscid portion of region II, (6) may be rewritten as

$$u^2(x,y) + v^2(x,y) \doteq (x_1/x)^2 . \tag{15}$$

From (1), (2), and (15), one obtains

$$-u(x_1,y) = \frac{\partial \psi(x_1,y)}{\partial y} = \left\{1 - [v(x_1,y)]^2\right\} = [1 - F(\psi)]^{1/2}, \tag{16}$$

where $0 \leq F(\psi) \leq 1$ for $0 \leq \psi \leq I$. Thus, $\psi(x_1, y)$ is given by

$$y = \int_0^{\psi(x_1, y)} \frac{d\mu}{\left\{1 - [F(\mu)]\right\}^{1/2}} \tag{17}$$

where $\psi \to I$ as $y \to \pi$ (cf. (10)). For $F(\psi) = \psi/I$,

$$\psi = y\left(1 - \frac{y}{2\pi}\right), \quad I = \frac{\pi}{2} \Rightarrow v^2(x_1, y) = \frac{2y}{\pi}\left(1 - \frac{y}{2\pi}\right). \tag{18}$$

For $F(\psi) = (\psi/I)^2$,

$$\psi = 2^{1/2}\sin(y/2), \quad I = 2^{1/2} \Rightarrow v^2(x_1, y) = \sin^2(y/2). \tag{19}$$

In fact,

$$v(x_1, y) \doteq 1 - \exp\left(-\frac{c}{\pi}y\right), \quad c \doteq 3, \tag{20}$$

might be more realistic, but is far less tractable.

In summary, if

$$\Psi(x, y) = \frac{\psi(x, y)}{I}, \quad G'(\Psi) = \frac{F'(\psi)}{I}, \tag{21}$$

Then the boundary-value problem is, for $0 \leq \Psi \leq 1$,

$$x\left(\frac{1}{x}\Psi_x\right) + \Psi_{yy} + G'(\Psi) = 0 ; \tag{22}$$

$$\Psi_x^2 + \Psi_y^2 = \left(\frac{x}{Ix_1}\right)^2 \text{ on } \Psi = 0, \quad \Psi_x^2 + \Psi_y^2 = 0 \text{ on } \Psi = 1; \tag{23a}$$

$\psi(x_1, y)$ given, with ψ periodic in y. (23b)

One might consider interchanging the roles of the dependent variable ψ and the independent variable x, and using perhaps some technique within the framework of the method of weighted residuals, perhaps with multigrid procedures (Brandt 1977), to extract the desired information from (21) - (23). However, first, alternative procedures are used to study (21) - (23).

3. TREATMENT OF A SEPARATED LAYER WITHOUT STRUCTURE

A momentum balance in an axisymmetric separated boundary layer without structure is examined. The thin sheet (or "eye wall") is the demarcation between any "eye," isobaric at pressure p_1^*, and a potential vortex with radially dependent pressure given by (12). At $r^* = r_1^*$, p^* in the sheet equals that in the "eye" (Fig. 2).

Henceforth in this section the symbol $r^*(z^*)$ denotes the outside surface of the "eye wall" without structure. At the point A, given by $\left[r^*(z^*), z^*\right]$, the principal radii of curvature are $\left\{1 + \left[r^{*\prime}(z^*)\right]^2\right\}^{3/2}/r^{*\prime\prime}(z^*)$, the radius of curvature in a plane containing the axis of symmetry and the streamline on which point A lies, and $r^*(z^*)\left\{1 + \left[r^{*\prime}(z^*)\right]^2\right\}^{1/2}$, the radius of curvature in a plane perpendicular to the streamline on which point A lies. Super prime denotes ordinary derivative with respect to the argument of the function. The velocity component in a plane containing the axis of symmetry is denoted q^*, while the velocity component (swirl) in a plane perpendicular to the axis of symmetry is denoted v^*. Thus, if $\hat{t}$ is a unit vector in the plane containing the axis, and $\hat{\theta}$ is perpendicular to $\hat{t}$ and refers to the azimuthal component in a cylindrical-polar-coordinate system, then

$$\vec{v}^* = q^*\hat{t} + v^*\hat{\theta}. \tag{24}$$

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The force balance perpendicular to the thin sheet at $[r^*(z^*), z^*]$ equates the pressure gradient consistent with a potential vortex outside the sheet, to the component of acceleration perpendicular to the sheet. There are two contributions to the acceleration, each involving the square of a velocity component over an appropriate radius of curvature:

$$- \frac{\partial}{\partial h^*} \left(\frac{p^*}{\rho_d^*} \right) = \frac{q^{*2} r^{*\prime\prime}}{[1 + (r^{*\prime})^2]^{3/2}} - \frac{v^{*2}}{r^*[1 + (r^{*\prime})]^{1/2}} \tag{25}$$

where h^* represents a coordinate running across the sheet.

Variations in velocity occurring across the sheet are not resolved $(\Gamma^* \equiv r_1^* v^*)$:

$$\int v^{*2} dh^* = \Gamma^{*2} \int \frac{dh^*}{[r^*(h^*)]^2} = r_1^{*2} v^{*2} \int \frac{dh^*}{[r^*(h^*)]^2} \equiv \frac{r_1^{*2} v^{*2} A^*}{r^{*2}} \Rightarrow$$

$$\frac{A^*}{r^{*2}} = \int \frac{dh^*}{[r^*(h^*)]^2} \quad ; \tag{26}$$

$$\int q^{*2} dh^* = v^{*2} \int \left[\frac{q^*(h^*)}{v^*} \right]^2 dh^* \equiv v^{*2} B^* \Rightarrow$$

$$B^* = \int \left[\frac{q^*(h^*)}{v^*} \right]^2 dh^* \quad . \tag{27}$$

The potential-vortex form of v^* is used in the definition of A^*.

From these definitions and from (12), to an accuracy that serves
current purposes,

$$\frac{v*^2}{2}\left(1 - \frac{r_1*^2}{r*^2}\right) = \frac{A*v*^2 r_1^2}{r*^3\left[1 + (r*')^2\right]^{1/2}} - \frac{B*v*^2 r*''}{\left[1 + (r*')^2\right]^{3/2}} . \tag{28}$$

The following nondimensionalization is introduced:

$$\bar{x} = \frac{r*}{r_1^*} \quad , \quad \bar{z} = \frac{z*}{r_1^*} \quad , \quad \alpha = \frac{2A*}{r_1^*} \quad , \quad \beta = \frac{2B*}{r_1^*} . \tag{29}$$

If $A* = (h*/2)$, $B* = (h*/2)$, seemingly reasonable values, then

$$\alpha \doteq \frac{h*}{r_1^*} \equiv h \quad , \qquad \beta \doteq \frac{h*}{r_1^*} \equiv h . \tag{30}$$

Under (29), (28) becomes (α,β specified)

$$1 - \frac{1}{\bar{x}^2} = \frac{\alpha}{\bar{x}^3(1 + \bar{x}'^2)^{1/2}} - \frac{\beta\bar{x}''}{(1 + \bar{x}'^2)^{3/2}} \quad \Rightarrow$$

$$\bar{x}'' = \frac{\alpha}{\beta}\frac{1 + \bar{x}'^2}{\bar{x}^3} - \frac{1}{\beta}\left(\frac{\bar{x}^2 - 1}{\bar{x}^2}\right)(1 + \bar{x}'^2)^{3/2} . \tag{31}$$

In this translationally invariant equation, it is taken that $\bar{x}(\bar{z} = 0) = 1$.
It is shown below that $\bar{x}$ is periodic in $\bar{z}$. Solution is sought for positive
and negative z, where the boundary conditions are

$$\bar{x}(0) = 1 \tag{32}$$

$$\bar{x}'(0) = \bar{x}_0', \text{ given const.} > 0. \tag{33}$$

The boundary conditions preclude odd or even solution for $\bar{x}$ in $\bar{z}$. Sought are $\bar{x}^+$, the largest value of $\bar{x}$, which occurs where $\bar{x}' = 0$, and $\bar{x}^-$, the smallest value of $\bar{x}$, which also occurs where $\bar{x}' = 0$. In that $\bar{x}^- < 1$, there is "overshoot" of the swirl, and, from (12), there is decrease of pressure from the value that holds in the "eye", i.e., in $0 < r^* < r^*(z^*)$; it is reiterated that the magnitude of the swirl overshoot and pressure decrease are of particular interest.

While numerical integration of (31) - (33) is required ultimately, some preliminary treatment is helpful. Since (31) is translationally invariant in $\bar{z}$, phase-plane analysis is introduced (Fig. 3):

$$\frac{d\bar{x}}{d\bar{z}} = \bar{P}(\bar{x}) \Rightarrow \frac{d^2\bar{x}}{d\bar{z}^2} = \frac{d\bar{P}(\bar{x})}{d\bar{z}} = \frac{d\bar{x}}{d\bar{z}}\frac{d\bar{P}}{d\bar{x}} = \bar{P}\frac{d\bar{P}}{d\bar{x}}. \tag{34}$$

Thus,

$$\frac{\beta\bar{P}\bar{P}'}{(1 + \bar{P}^2)^{3/2}} = \frac{\alpha}{\bar{x}^3(1 + \bar{P}^2)^{1/2}} + \frac{1}{\bar{x}^2} - 1. \tag{35}$$

The slope is infinite at $\bar{P} = 0$ and is zero where

$$\frac{\alpha}{\bar{x}^3(1 + \bar{P}^2)^{1/2}} + \frac{1}{\bar{x}^2} - 1 = 0; \tag{36}$$

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the intersection of the curve of (36) with the $\overline{P} = 0$ ray is given by

$$\frac{\alpha}{\overline{x}_*^3} + \frac{1}{\overline{x}_*^2} - 1 = 0 , \tag{37}$$

where $\overline{x}_* = 1$ for $\alpha = 0$, $\overline{x}_* > 1$ for $\alpha > 0$. For $1 \gg \alpha > 0$, $\overline{x}_* = 1 + (\alpha/2) - (3\alpha^2/8) + \ldots$.

Although the following development is not pursued to the extent of obtaining results, it may be worth noting that if

$$H(\overline{x}) = (1 + \overline{P}^2)^{-1/2} , \tag{38}$$

then (35) becomes

$$-\beta H' = \frac{\alpha H}{\overline{x}^3} + \frac{1}{\overline{x}^2} - 1 . \tag{39}$$

If

$$\tau = \overline{x}^{-2} , \tag{40}$$

then

$$2\beta \frac{dH}{d\tau} - \alpha H = \tau^{-1/2} - \tau^{-3/2} , \tag{41}$$

or

$$H = (\beta \tau^{1/2})^{-1} - \left(\frac{\pi}{2\alpha\beta}\right)^{1/2} \left(1 + \frac{\alpha}{\beta}\right) \exp\left(\frac{\alpha\tau}{2\beta}\right) \text{erfc}\left(\frac{\alpha\tau}{2\beta}\right)^{1/2}$$

$$+ E \exp\left(\frac{\alpha\tau}{2\beta}\right) , \tag{42}$$

where E is a const. of integration. From (38), (40), and the definition $\overline{P} \equiv (d\overline{x}/d\overline{z})$, one may write formally

$$\frac{d\overline{x}}{d\overline{z}} = \left\{1 + [H(\tau)]^{-2}\right\}^{1/2} = \frac{d(\tau^{-1/2})}{d\overline{z}} \, . \tag{43}$$

Thus,

$$d\overline{z} = \frac{d(\tau^{-1/2})}{\left\{1 + [H(\tau)]^{-2}\right\}^{1/2}} \, , \tag{44}$$

where $H(\tau)$ is given by (42).

For the special case $\alpha = 0$, i.e., no swirl, multiplication of (35) by $\overline{x}'$ yields

$$\frac{\beta}{(1 + \overline{x}'^2)^{1/2}} = \frac{(\overline{x} - 1)^2}{\overline{x}} + \frac{\beta}{m} \, , \tag{45}$$

where the constant of integration has been written as (β/m), with

$$m \equiv (1 + \overline{x}_0'^2)^{1/2} \tag{46}$$

for consistency. It is recalled that $\overline{x}_0' (\equiv d\overline{x}(0)/d\overline{z})$ is a given positive finite constant. Inspection of (45) reveals that $\overline{x}'$ is maximum at $\overline{x} = 1$, so $(1 + \overline{x}_0'^2)^{1/2}$ is the maximum value of $(1 + \overline{x}'^2)^{1/2}$, whence the symbol m. At $\overline{x}' = 0$,

$$\overline{x}^{\pm} = 1 + \frac{\beta}{2}\left(1 - \frac{1}{m}\right) \pm \left[\frac{\beta}{2}\left(1 - \frac{1}{m}\right)\right]^{1/2} \left[2 + \frac{\beta}{2}\left(1 - \frac{1}{m}\right)\right]^{1/2} \tag{47}$$

For $\overline{x}_0' \to 0$ so $m \to 1$, $\overline{x}^{\pm}$ merge to unity; this case involves a vertically separating surface inflow layer, and hence no overshoot. For $\overline{x}_0' \to \infty$ so $m \to \infty$, $\overline{x}^{\pm}$ remain bounded:

$$\bar{x}^{\pm} = 1 + \frac{\beta}{2} \pm \left(\frac{\beta}{2}\right)^{1/2} \left[2 + \frac{\beta}{2}\right]^{1/2} ; \tag{48}$$

this case involves effectively horizontal inflow of the separating surface inflow layer, and leads to the minimum value of $\bar{x}^-$ for fixed β. For $\beta = 0.1$, $\bar{x}^+ \doteq 1.37$ and $\bar{x}^- \doteq 0.73$; for $\beta = 0.2$, $\bar{x}^+ \doteq 1.56$ and $\bar{x}^- \doteq 0.64$. The plausible range of $\beta = 0(0.2)$, from a crude analysis of the turnaround to be published separately; it is recalled that β is associated with in-plane motion. These results suggest what proves to be a general trend: $\bar{x}^-$ decreases as $\bar{x}'_0$ and β increases.

Finite values for α indicate finite swirl; increasing α yields larger $\bar{x}^-$ and smaller overshoot. From results of numerical integration, for finite α, $\bar{x}^{\pm}$ approach finite values as $\bar{x}'_0 \to \infty$ for fixed β. For the plausible values $\alpha = \beta = 0.2$, for $\bar{x}'_0 = 2, 5, 10$, the corresponding values of $\bar{x}^- = 0.77$, 0.72, 0.70. Hence, swirl overshoots in the range of about 10% seem plausible, but no more; estimates that swirl in the turnaround exceeds the swirl at $\bar{x} = 1$, $\bar{z} = h$ (in terms of the definitions of (29)) by about 100% (Lewellen 1977) are excessive according to this analysis. Further results are given in Table 1 and graphical presentation is given in Figures 4 and 5.

4. REFORMATION OF THE BOUNDARY-VALUE PROBLEM

A change of variables from (r^*, z^*) to (ψ^*, Θ), where the angle Θ gives the orientation in the (r^*, z^*) plane of the streamline relative to a line parallel to $z^* = 0$ (Fig. 6), is developed for the boundary-value problem introduced in Section 2. The restriction to constant-density flow is relaxed.

The streamfunction ψ^* is redefined by

$$\rho^* q^* r^* \sin\Theta = \frac{\partial \psi^*}{\partial r^*} , \tag{49}$$

$$\rho^* q^* r^* \cos\Theta = \frac{\partial \psi^*}{\partial z^*} . \tag{50}$$

Here r^* again denotes the conventional cylindrical radial coordinate at which the velocity is to be calculated, q^* is the velocity in the (r^*, z^*) plane, and n^* (introduced below) is the distance normal to the streamline.

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The quantity $q^* \sin\Theta$ is the vertical velocity w^*, and the quantity $-q^* \cos$ is the radial velocity u^*.

The change of variables from (r^*, z^*) to (ψ^*, Θ) requires that both $(\partial/\partial r^*)_{z^*}$ and $(\partial/\partial z^*)_{r^*}$ be expressed in terms of $(\partial/\partial \psi^*)_\Theta$ and $(\partial/\partial \Theta)_{\psi^*}$, where the subscript is held fixed. Consider a function f^* that depends on r^* and z^*, and, therefore, on ψ^* and Θ. One may write

$$df^* = \left(\frac{\partial f^*}{\partial \psi^*}\right)_\Theta d\psi^* + \left(\frac{\partial f^*}{\partial \Theta}\right)_{\psi^*} d\Theta$$

$$= \left(\frac{\partial f^*}{\partial \psi^*}\right)_\Theta \left[\left(\frac{\partial \psi^*}{\partial r^*}\right)_{z^*} dr^* + \left(\frac{\partial \psi^*}{\partial z^*}\right) dz^*\right] + \left(\frac{\partial f^*}{\partial \Theta}\right)_{\psi^*} d\Theta$$

$$= \left(\frac{\partial f^*}{\partial \psi^*}\right)_\Theta \left[\rho^* q^* r^* \sin\Theta\, dr^* + \rho^* q^* r^* \cos\Theta\, dz^*\right] + \left(\frac{\partial f^*}{\partial \Theta}\right)_{r^*} d\Theta. \quad (51)$$

Henceforth, the quantity being held fixed is no longer explicitly written, for brevity.

The usual transformation rules apply, i.e.,

$$\Theta_{z^*} = \frac{r^*_{\psi^*}}{J^*}, \quad \Theta_{r^*} = -\frac{z^*_{\psi^*}}{J^*}, \quad \psi^*_{z^*} = -\frac{r^*_\Theta}{J^*}, \quad \psi^*_{r^*} = \frac{z^*_\Theta}{J^*},$$

$$(52a,b,c,d)$$

where

$$J^* = r^*_{\psi^*} z^*_\Theta - r^*_\Theta \psi^*_{z^*}, \quad (52e)$$

and subscript here denotes partial differentiation.

Division of (51) by dz^* and letting $dz^* \to 0$ as dr^* is held fixed at zero, one obtains

$$f^*_{z^*} = f^*_{\psi^*} \rho^* q^* r^* \cos\Theta + f^*_\Theta \Theta_{z^*}; \quad (53)$$

similarly,

$$f^*_{r^*} = f^*_{\psi^*} \rho^* q^* r^* \sin\Theta + f^*_\Theta \Theta_{r^*}. \quad (54)$$

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From (52), (53), and (54),

$$f^*_{z*} = \rho^* q^* r^* \cos\Theta \, f^*_{\psi*} + \frac{r^*_{\psi*}}{r^*_{\psi*} z^*_\Theta - r^*_\Theta z^*_{\psi*}} \, f^*_\Theta \; ; \qquad (55)$$

$$f^*_{r*} = \rho^* q^* r^* \sin\Theta \, f^*_{\psi*} - \frac{z^*_{\psi*}}{r^*_{\psi*} z^*_\Theta - r^*_\Theta z^*_{\psi*}} \, f^*_\Theta \; . \qquad (56)$$

In particular, the directional derivative _in_ the r*,z* frame, with direction normal to a constant ψ^* line, is

$$f^*_{n*} = f^*_{r*} \sin\Theta + f^*_{z*} \cos\Theta \; ;$$

i.e.,

$$f^*_{n*} = \rho^* q^* r^* \, f^*_{\psi*} + \frac{r^*_{\psi*} \cos\Theta - z^*_{\psi*} \sin\Theta}{r^*_{\psi*} z^*_\Theta - r^*_\Theta z^*_{\psi*}} \, f^*_\Theta \; . \qquad (57)$$

If one adds (55) with f* replaced by z* to (56) with f* replaced by r*, one gets

$$z^*_{\psi*} \cos\Theta + r^*_{\psi*} \sin\Theta = \frac{1}{\rho^* q^* r^*} \; . \qquad (59)$$

The quotient (4d) ÷ (4c) gives

$$z^*_\Theta = - r^*_\Theta \tan\Theta \; . \qquad (59)$$

Also, by use of (58) and (59) to eliminate z* in the expression for J* given in (52e),

$$J^* = -r^*_{\psi*} r^*_\Theta \tan\Theta - r^*_\Theta \left(\frac{1}{\rho^* q^* r^* \cos\Theta} - r^*_{\psi*} \tan\Theta \right)$$

$$= \frac{-r^*_\Theta}{\rho^* q^* r^* \cos\Theta} \; . \qquad (60)$$

Furthermore,

$$r^*_{\psi*} \cos\Theta - z^*_{\psi*} \sin\Theta = r^*_{\psi*} \cos\Theta - \left[\frac{1}{\rho^* q^* r^* \cos\Theta} - r^*_{\psi*} \tan\Theta \right] \sin\Theta$$

$$= \frac{r^*_{\psi*}}{\cos\Theta} - \frac{\tan\Theta}{\rho^* q^* r^*} \; . \qquad (61)$$

Substitution of (60) and (61) in (57) gives

$$f^{\star}{}_{n^{\star}} = \rho^{\star}q^{\star}r^{\star}f^{\star}{}_{\psi^{\star}} - \frac{\dfrac{r^{\star}{}_{\psi^{\star}}}{\cos\Theta} - \dfrac{\tan\Theta}{\rho^{\star}q^{\star}r^{\star}}}{r^{\star}{}_{\Theta}} \, \rho^{\star}q^{\star}r^{\star} \, \cos\Theta \, f^{\star}{}_{\Theta}$$

$$= \rho^{\star}q^{\star}r^{\star}f^{\star}{}_{\psi^{\star}} - \rho^{\star}q^{\star}r^{\star} \frac{r^{\star}{}_{\psi^{\star}}}{r^{\star}{}_{\Theta}} \, f^{\star}{}_{\Theta} + \frac{\sin\Theta}{r^{\star}{}_{\Theta}} \, f^{\star}{}_{\Theta} \ .$$

Thus, the conservation of momentum in the direction normal to a surface $\psi^{\star}$ = const. is given by

$$\rho^{\star}q^{\star}r^{\star}p^{\star}{}_{\psi^{\star}} - \rho^{\star}q^{\star}r^{\star} \frac{r^{\star}{}_{\psi^{\star}}}{r^{\star}{}_{\Theta}} \, p^{\star}{}_{\Theta} + \frac{\sin\Theta}{r^{\star}{}_{\Theta}} \, p^{\star}{}_{\Theta}$$

$$= \frac{\rho^{\star}v^{\star 2}\sin\Theta}{r^{\star}} + \frac{\rho^{\star}q^{\star 2}\cos\Theta}{r^{\star}{}_{\Theta}} - \rho^{\star}g^{\star} \, \cos\Theta \ .$$

[The middle term on the right-hand side is obtained thus:

$$- \rho^{\star}q^{\star 2} \, \frac{\partial\Theta}{\partial s^{\star}} = - \rho^{\star}q^{\star 2} \, \frac{1}{(\partial s^{\star}/\partial\Theta)} = + \rho^{\star}q^{\star 2} \, \frac{\cos\Theta}{(\partial r^{\star}/\partial\Theta)} \ ,$$

where $s^{\star}$ is distance along a streamline. The principal curvatures of a surface $\psi^{\star}$ = const. in the $(r^{\star}, z^{\star})$ plane are thus seen to be $(\partial\Theta/\partial s^{\star})$ and $\sin\Theta/r^{\star}$.] Rearrangement gives

$$\left(p^{\star}{}_{\psi^{\star}} - \frac{v^{\star 2}\sin\Theta}{r^{\star 2}q^{\star}} + \frac{g^{\star}\cos\Theta}{q^{\star}r^{\star}} \right) r^{\star}{}_{\Theta} - \left(r^{\star}{}_{\psi^{\star}} - \frac{\sin\Theta}{\rho^{\star}q^{\star}r^{\star}} \right) p^{\star}{}_{\Theta}$$

$$= \frac{q^{\star}}{r^{\star}} \, \cos\Theta \ . \tag{62}$$

If one divides (58) by $\cos\Theta$, differentiates the result with respect to Θ, and then subtracts from this result the derivative of (59) with respect to $\psi^{\star}$, one obtains

$$r^{\star}{}_{\psi^{\star}} = \cos^{2}\Theta \left(\frac{1}{\rho^{\star}q^{\star}r^{\star} \, \cos\Theta} \right)_{\Theta} \ . \tag{63}$$

The pertinent equations include an equation of state, a (pseudo) adiabatic relation[#], conservation of angular momentum

$$r^* v^* = F^*(\psi^*) \ , \tag{64}$$

and Bernoulli's equation

$$\int \frac{dp^*}{\rho^*} + \frac{q^{*2}}{2} + \frac{v^{*2}}{2} + g^* z^* = G^*(\psi^*) \ , \tag{65}$$

where $F^*(\psi^*)$, $G^*(\psi^*)$ are implied by the boundary conditions and are taken as specified. Equations (59), (62), and (63) supplement this list. There are seven equations for the seven unknowns q^*, v^*, r^*, z^*, p^*, ρ^*, and T^*. Each unknown is a function of ψ^* and Θ.

The appropriate initial conditions are

$$\rho^* = \rho_1^* \ , \ r^* = r_1^* \ , \ F^*(\psi^*) \text{ given}, \ G^*(\psi^*) \text{ given on } \Theta = \Theta_0 = 0,$$

$$0 < \psi^* < \psi_{max}^* \ . \tag{66}$$

At $\Theta = \Theta_0 = 0$, $0 < \psi^* < \psi_{max}^*$, one anticipates, for $r^* = r_1^*$,

$$q^{*2} + v^{*2} \doteq V^{*2}, \tag{67a}$$

$$p_1^* - p^* \doteq \rho_1^* g^* z^* \ . \tag{67b}$$

The boundary conditions <u>for small</u> z^* are

$$p^*(\psi^* = 0, \Theta) \doteq p_1^* \ , \tag{68}$$

$$p^*\left(\psi_{max}^*, \ \Theta\right) - p_1^* \doteq - \frac{\rho_1^*}{2} \left[v^{*2}\left(\psi_{max}^*, \ \Theta\right) - V^{*2} \right]$$

$$= - \frac{\rho_1^* \ \Gamma^{*2}}{2} \left[\frac{1}{r^{*2}\left(\psi_{max}^*, \ \Theta\right)} - \frac{1}{r_1^{*2}} \right] \ . \tag{69}$$

[#] The reference here is to a relation among the thermodynamic variables *obtained from the second law of thermodynamics.* In the absence of condensation of water vapor, the dry-adiabatic relation suffices.

All quantities with subscript unity are given constants, ψ^*_{max} is calculable, and $\Gamma^* = r_1^* V^*$, with V^* given.

In Figure 6, ρ_1^* is denoted ρ^*_{d*}, and the <u>possibility</u> that the boundary-value problem possesses a reflection-type solution at low altitudes is explored schematically.

With the boundary-value problem now displayed in (ψ^*, Θ) coordinates, it is moot whether this formulation is preferable to that in (r^*, z^*) coordinates.

5. GENERALIZATION TO A VARIABLE-DENSITY MODEL--FURTHER COMMENTS

For a formulation valid through the depth of the tropopause, it is useful to review tenets of the modeling at this point, since greater detail concerning the thermodynamics is to be added.

When the state of the ambient atmosphere is known, and when the circulation Γ^* of the potential motion is given, one can calculate the inviscid, quasisteady motion which is consistent with any specified central-core pressure distribution, the logical extremes of which are (1) the dry-adiabatic condition consistent with the total enthalpy of the ambient air at sea-level, and (2) something weaker than the moist-adiabat corresponding to that same sea-level condition.

The general character of the configuration is given in Figure 7. In region I any radial and/or vertical motion is supposed to be so small that it is ignorable in the dynamic balance. The angular momentum Γ^* is pre-scribed, most simply as a constant; ultimately, interest centers on the case where Γ^* is a function of z^*. Region II is the boundary layer which transports swirling fluid inward; r_1^* is the radius at the pressure in the potential vortex at ground vortex is the same as the prescribed pressure at $r^* = 0$, $z^* = 0$. Region III contains updraft fluid and its state trajec-tory is <u>dynamically</u> moist adiabatic; i.e., the total enthalpy is constant. Region IV is nearly stagnant and contains either updraft air (nearly moist adiabatic) or recompressed air from z_o^*, the altitude at which, in the ambient, the total enthalpy is the same as that at ground level. Typically, that ambient enthalpy is of the character depicted in Figure 8.

It has already been noted in Section 4 that the boundaries of region III are not known; the coordinates r^* and z^* are not particularly well suited to a successful analysis. On the other hand, if one defines a streamfunction ψ^* and a "velocity direction Θ", the boundaries are fixed in the (ψ^*, Θ) plane and it may be useful to formulate the dynamic balances in that framework.

In writing the equations for the conservation of momentum for the inviscid flow of an air, water-vapor mixture, one usefully adopts the hypothesis that no supercooling occurs, but that once the temperature drops far enough to imply saturation, water condenses at precisely the rate needed to keep the vapor at the equilibrium saturation level.

The angular momentum of each particle is conserved, and the total head is preserved on each streamline. Momentum conservation normal to a streamline and geometric relations that supplement these statements were derived in Section 4.

Finding the locus of thermodynamic states that characterize the moist adiabat based on sea-level ambient conditions; assignment of an altitude to the tropopause, denoted z_o^* in Fig. 8; and determination of the pressure distribution with height in region IV, denoted $p_o^*(z^*)$ in Fig. 7 -- these steps are standard (Fendell 1974).

Finding the pressure in region I compatible with the ambient atmosphere, $p_1^*(r^*, z^*)$, is now discussed. If one discards the subscript unity,

$$p^*_{r^*} = \rho^* v^{*2}/r^* ,$$

$$p^*_{z^*} = -\rho^* g^* ;$$

hence,

$$g^* p^*_{r^*} = -\frac{v^{*2}}{r^*} p^*_{z^*} = -\frac{\Gamma^{*2}(z^*)}{r^{*3}} p^*_{z^*} .$$

If one seeks solution in the form

$$p^*(n^*) = p^*\left[m^*(z^*) - s^*(r^*)\right] ,$$

one obtains (if prime denotes the ordinary derivative)

$$p^{*\prime}(\eta^*)\left[g^* s^{*\prime}(r^*)r^{*3} - \Gamma^{*2}(z^*)m^{*\prime}(z^*)\right] = 0 \; ;$$

i.e., without loss of generality,

$$s^*(r^*) = -\frac{1}{2g^* r^{*2}} \; ,$$

$$m^*(z^*) = \int_0^{z^*} \frac{dz'}{\Gamma^{*2}(z')} \; .$$

For the special case of a vortex in region I that is invariant with altitude, $\Gamma^* = \text{const.}$,

$$m^*(z^*) = z^*/\Gamma^{*2} \; ;$$

the choice of $p^*(\eta^*)$ which matches the ambient atmosphere at $r^{*-2} \ll 1$ is

$$p^* = p^*_{ambient}\left[z^* + \Gamma^{*2}/(2g^* r^{*2})\right] \; .$$

[In this approximation the ambient formally is taken to hold at an infinite radial distance, but the discrepancy from a match to an ambient at a value of (r^*_0/r^*_1) of only ten results in an error of about merely one percent.]

For the somewhat more plausible case of a vortex, the angular momentum of which decreases linearly from a value of Γ^*_0 at sea-level to zero at the tropopause $z^* = z^*_0$,

$$\eta^* = \frac{z^*}{z^*_0 - z^*} + \frac{\Gamma^{*2}_0}{2g^* r^{*2} z^*_0} \; ,$$

such that at $r^* \to \infty$,

$$z^* = \frac{z^*_0 \eta^*}{\eta^* + 1} \; .$$

Thus a solution of the first-order partial differential equation $p^*(\eta^*)$, any function of η^*, also satisfies the boundary condition,

$p*(n*) \longrightarrow p^*_{ambient}(z*)$ for $r* \longrightarrow \infty$, if one takes as the function

$$p*(n*) = p^*_{ambient}\left[\frac{z* + \dfrac{\Gamma^{*2}_0(z^*_0 - z*)}{2g*r*^2 z^*_0}}{1 + \dfrac{\Gamma^{*2}_0(z^*_0 - z*)}{2g*r*^2 z^{*2}_0}}\right].$$

In region III the pressure on the streamline contiguous to region IV must match the specified $p^*_0(z*)$ in the core, and the pressure on the streamline contiguous to region I must match the pressure in the potential vortex $p^*_{ambient}[m^{*-1}(m*(z*) - s*(r*))]$. These statements constitute the boundary conditions. At $r = r^*_1$, $0 < z* < h*$, the initial value of each dependent variable is specified on the basis of the known solution for region II.

Obtaining the solution of this formulation of "eye wall" with structure, throughout the depth of the troposphere, is the next goal of this on-going investigation.

Table 1. Numerical Results for Extrema of the Turnaround, from Integration of the Initial-Value Problem (31)-(33).

α	β	α/β	$\bar{x}'(0) \equiv \bar{x}_0'$	$(\bar{x}^-)_{calc}$	$(\bar{x}^-)_{\alpha=0}$	$(\bar{x}^+)_{calc}$	$(\bar{x}^+)_{\alpha=0}$
0.1	0.1	0.0	2.0	0.791	0.789	1.264	1.266
0.0	0.1	0.0	5.0	0.754	0.751	1.326	1.329
0.0	0.1	0.0	10.0	0.742	0.738	1.348	1.352
0.0	0.1	0.1	100.0	0.732	0.727	1.366	1.372
0.0	0.2	0.0	2.0	0.718	0.714	1.392	1.397
0.0	0.2	0.0	5.0	0.671	0.664	1.489	1.497
0.0	0.2	0.0	10.0	0.656	0.647	1.524	1.533
0.0	0.2	0.0	100.0	0.644	0.632	1.553	1.565
0.1	0.1	1.0	2.0	0.820	0.789	1.295	1.266
0.1	0.1	1.0	5.0	0.777	0.751	1.348	1.329
0.1	0.1	1.0	10.0	0.762	0.738	1.366	1.352
0.1	0.1	1.0	100.0	0.750	0.727	1.381	1.372
0.1	0.2	0.5	2.0	0.748	0.714	1.422	1.397
0.1	0.2	0.5	5.0	0.695	0.664	1.510	1.497
0.1	0.2	0.5	10.0	0.677	0.647	1.541	1.533
0.1	0.2	0.5	100.0	0.663	0.632	1.567	1.566
0.1	0.5	2.0	2.0	0.876	0.846	1.212	1.181
0.1	0.5	2.0	5.0	0.841	0.818	1.244	1.223
0.1	0.5	2.0	10.0	0.828	0.808	1.254	1.237
0.1	0.5	2.0	100.0	0.818	0.800	1.264	1.250
0.2	0.1	2.0	2.0	0.845	0.789	1.325	1.266
0.2	0.1	2.0	5.0	0.796	0.751	1.369	1.329
0.2	0.1	2.0	10.0	0.779	0.738	1.384	1.352
0.2	0.1	2.0	100.0	0.765	0.727	1.396	1.372
0.2	0.2	1.0	2.0	0.774	0.714	1.452	1.397
0.2	0.2	1.0	5.0	0.716	0.664	1.530	1.497
0.2	0.2	1.0	10.0	0.695	0.647	1.558	1.533
0.2	0.2	1.0	100.0	0.678	0.632	1.581	1.566
0.2	0.4	0.5	2.0	0.685	0.615	1.652	1.606
0.2	0.4	0.5	5.0	0.617	0.550	1.789	1.772
0.2	0.4	0.5	10.0	0.594	0.528	1.838	1.832
0.2	0.4	0.5	100.0	0.578	0.509	1.877	1.887

Notes: Results for $(\bar{x}^\pm)_{\alpha=0}$ are from (40). Also, $\bar{x}$ for $\bar{x}_0' = 100$ are within 2% of values for $\bar{x}_0' \to \infty$, for α, β studied.
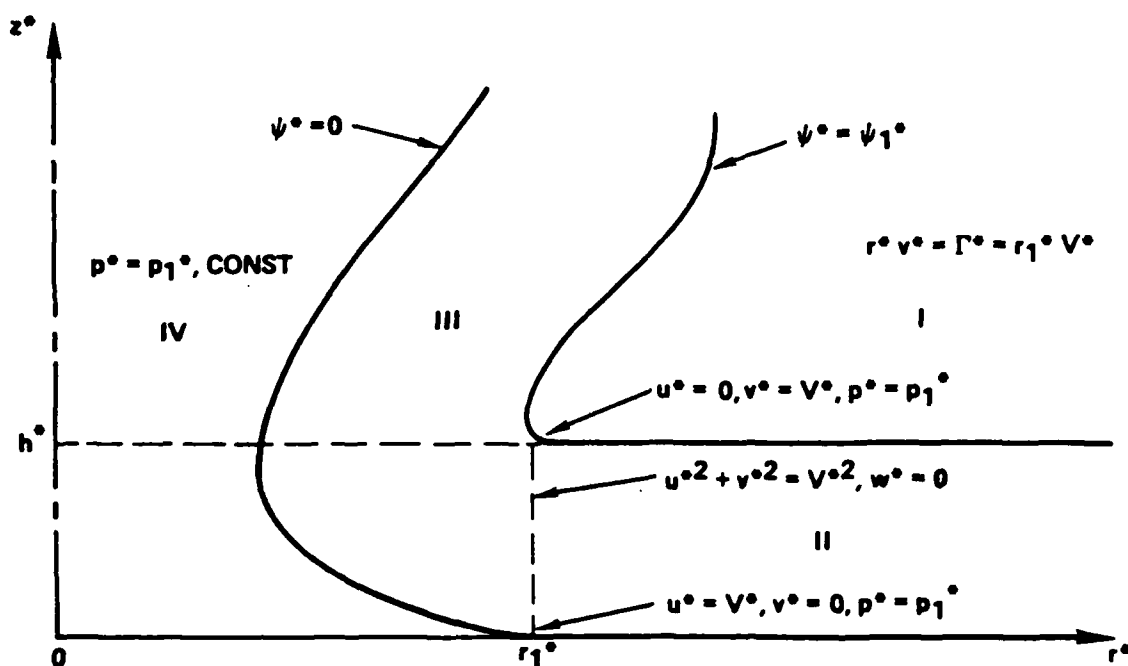
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Figure 1. Schematic, not to scale, of an inviscid model of the turnaround region of a severe vertical quasisteady axisymmetric vortex. The "eye", region IV, is isobaric at pressure p_1^*, over the vertical extent of interest here, where p_1^* is also the pressure at $r^* = r_1^*$, $z^* = h^*$ (because the pressure is approximately invariant across the surface inflow layer II). The "eye wall", region III, is demarcated by two streamsurfaces, $\psi^*(r^*,z^*) = $ const. (the position of each to be determined), encompassing the mass efflux from region II. It is recalled that a potential vortex holds in region I.
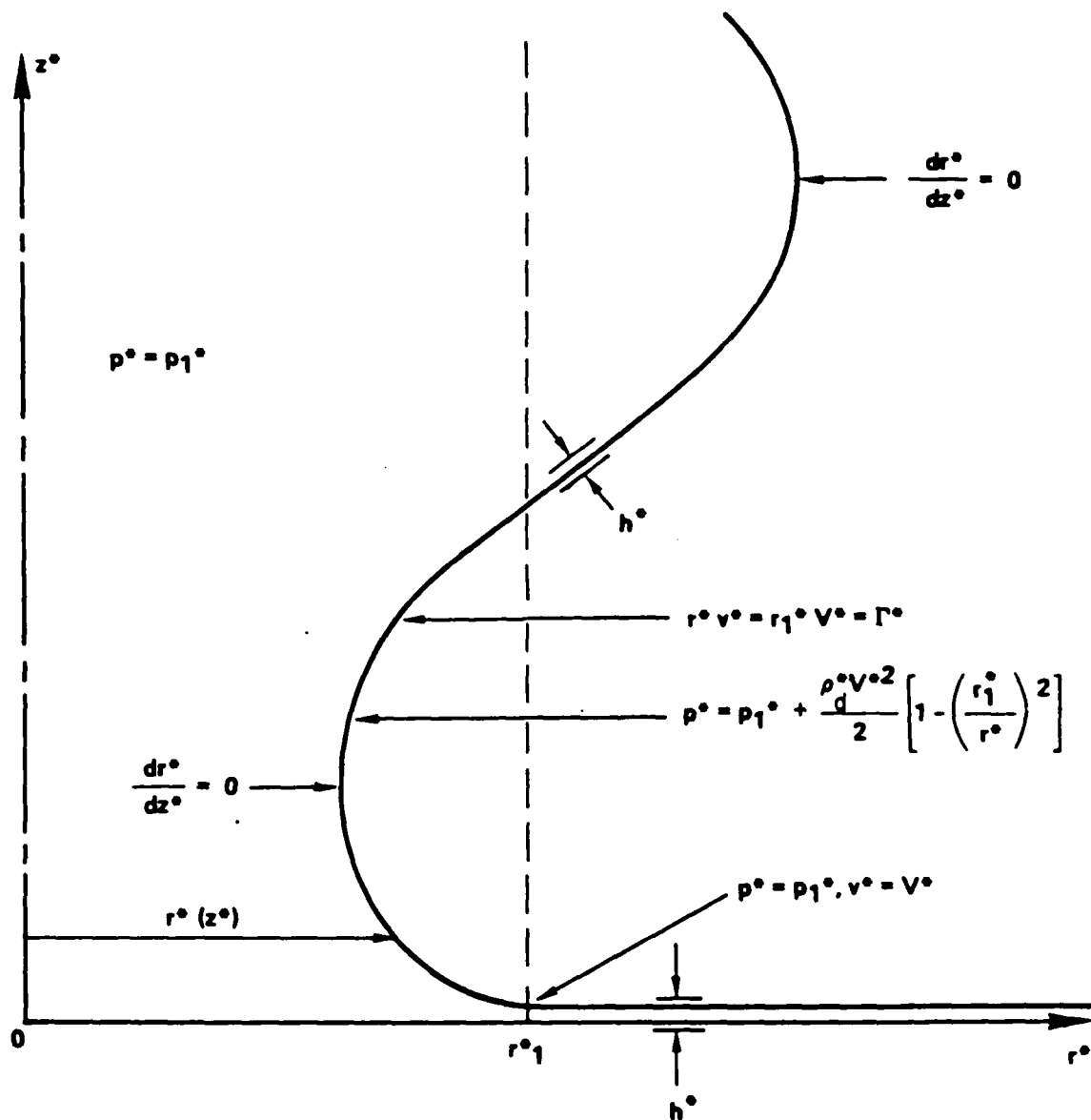
Figure 2. Schematic of the location of an inviscid "eye wall without structure" demarcating the interface between an isobaric non-swirling "eye" at pressure p_1^* and a potential vortex. The sheet representing the "eye wall" has thickness $h^* \to 0$; in the turn-around region, its displacement from the axis, as a function of height above the ground plane, is denoted $r^*(z^*)$.
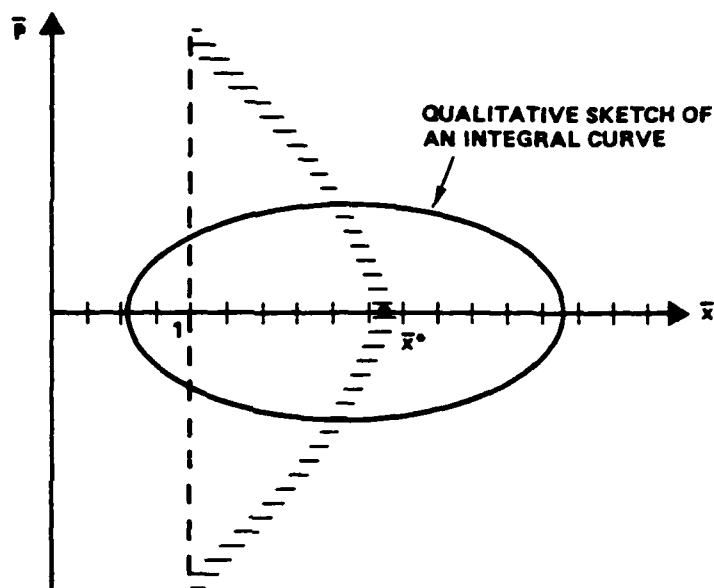
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Figure 3. Phase-plane properties of equation (35), where $\bar{P} = (d\bar{x}/d\bar{z})$, with $\bar{x}$ the dependent variable ($\bar{x} > 0$) and $\bar{z}$ the independent variable. Isoclines of zero and infinite slope are noted. At $\bar{x} = 1$, the distance from the axis of symmetry at which the surface inflow layer separates, a finite positive slope is adopted. For $\alpha > 0$, $\bar{x}_* > 1$; for $\alpha = 0$, $\bar{x}_* = 1$. The sketched trajectory (solution curve) is a limit cycle (closed curve indicative of periodic behavior). The periodicity is not of physical interest.
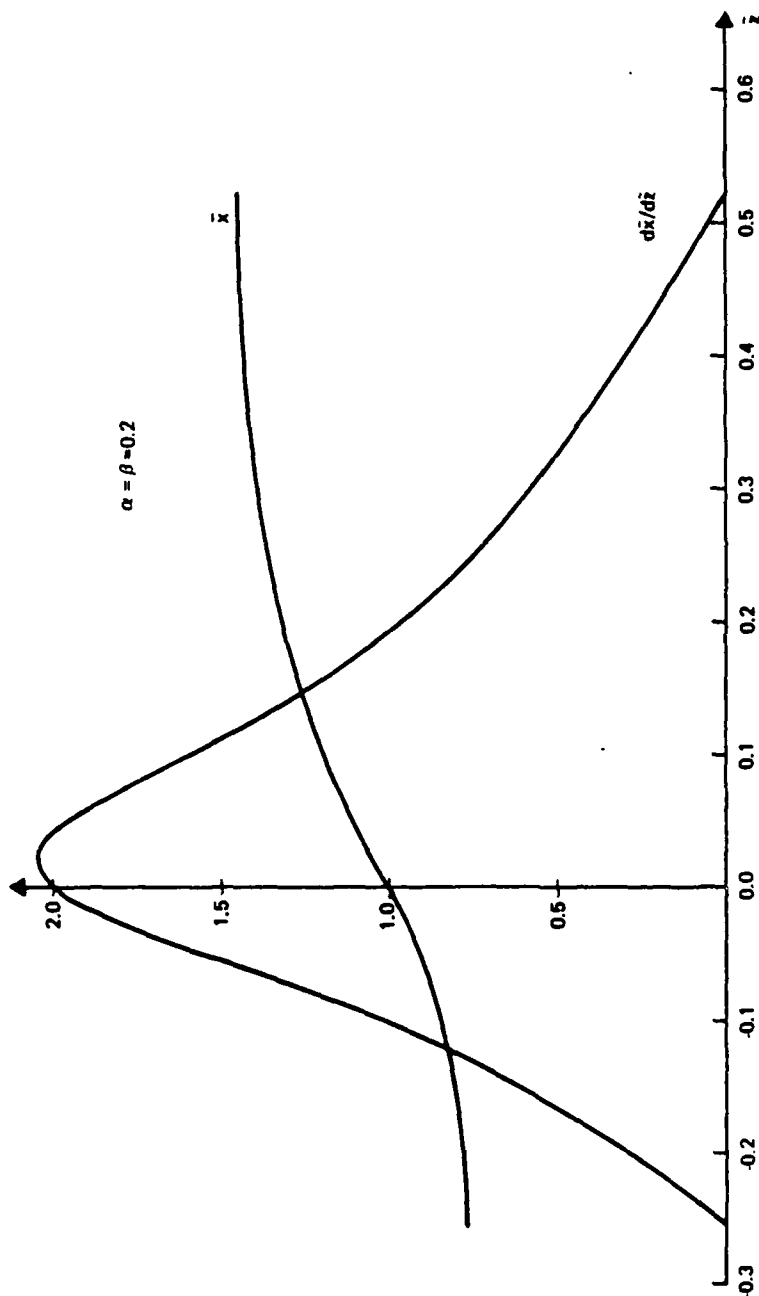
Figure 4. The solution to the boundary-value problem posed by equations (32), (33), and (35), for $\overline{x}_0' = 2$; the solution may be completed by symmetry considerations to constitute a full cycle.
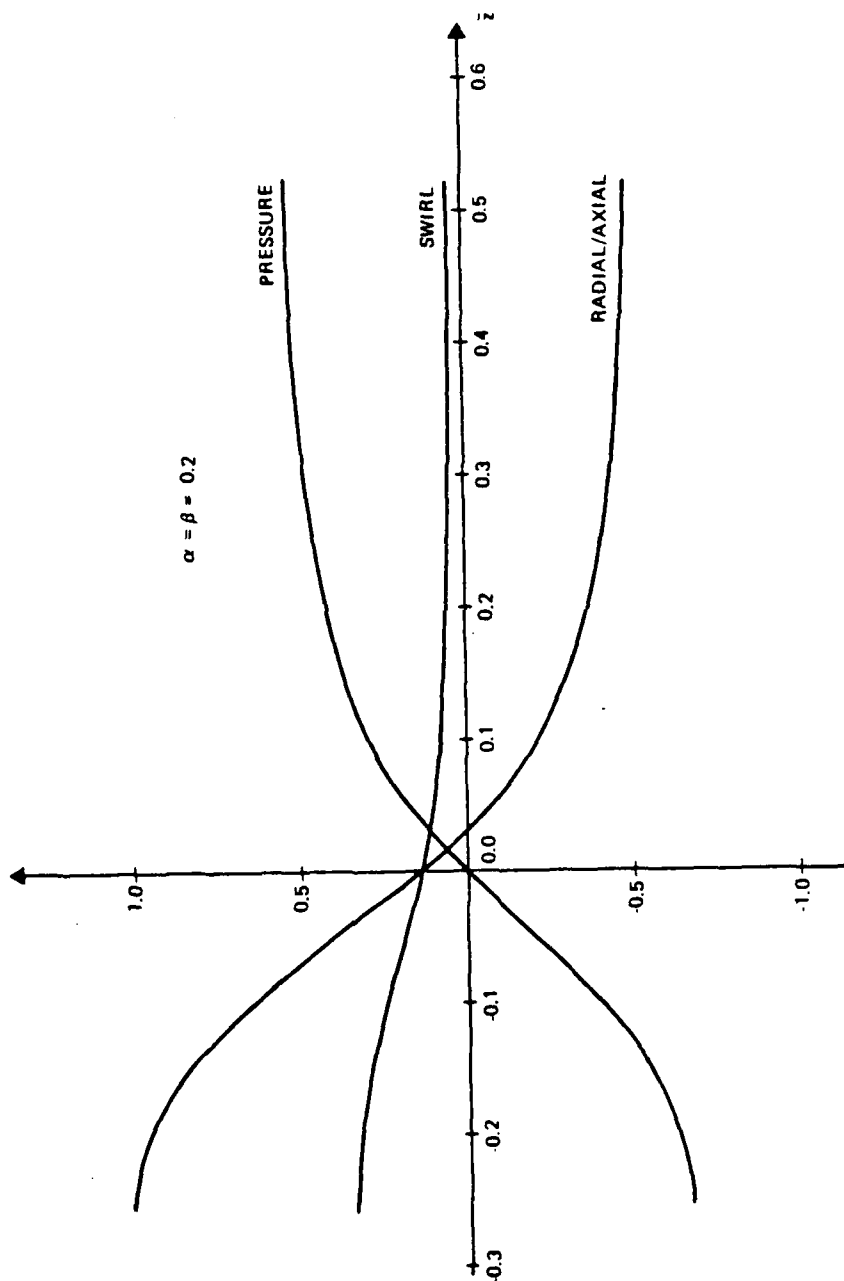
Figure 5. The solution to the boundary-value problem posed by equations (32), (33), and (35), for $\alpha = 0.2$, $\beta = 0.2$, and $\bar{x}_0' = 2$, given in Figure 10, is further discussed. The "pressure" denotes the value of $(1 - \bar{x}^{-2})$; the "swirl" denotes the value of $\bar{\alpha}\bar{x}^{-3}[1 + \bar{x}^2]^{-1/2}$; the "radial/axial" denotes the value of "swirl" minus "pressure", i.e., the left-hand side of (35).
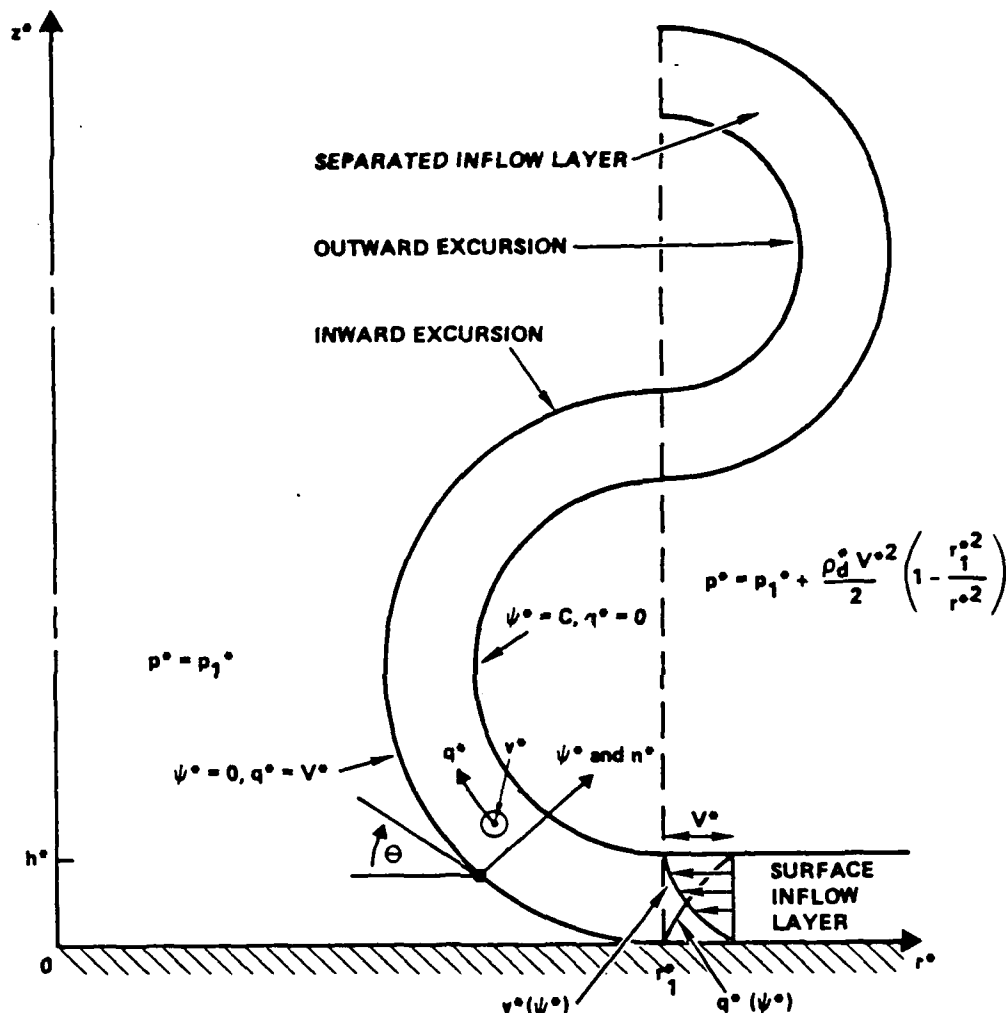
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Figure 6. This schematic of the inward, then outward, excursion of the separated inflow layer with structure, indicates the bounding stream surfaces $\psi^* = 0$ and $\psi^* = C$. The in-plane radial-axial flow speed is denoted q^*; the swirl about the vertical axis, v^*. The inward excursion need be solved for streamfunction-inclination angle $\theta = 0$ to $\theta = (\pi/2)$, since the solution for $\theta = (\pi/2)$ to $\theta = \pi$ may then be obtained by reflection. An analogous statement holds for the outward excursion. The "eye" is isobaric at pressure p_1^*, but the potential-vortex pressure varies with radial position r^* as noted.
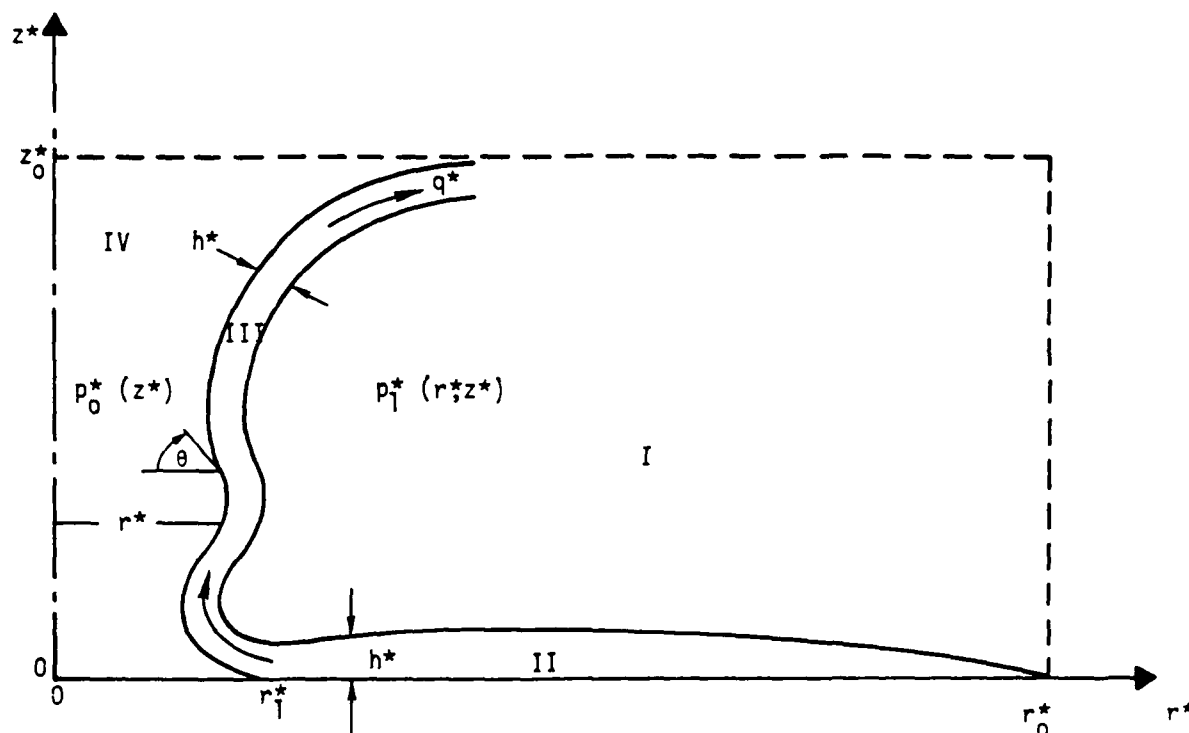
Figure 7. Another schematic of the postulated four-part model of the structure of a mature intense hurricane, of radial extent r_o^* and axial extent z_o^* (the tropopause). The potential vortex in region I is here ascribed modest axial variation, such that the associated pressure field is denoted $p_1^*(r^*,z^*)$. The low-level swirling influx (region II) grows to thickness h^*, and separates to form the "eye wall" (region III); the flow speed independent of swirling is denoted q^*, and the inclination of the secondary (i.e., radial and axial) flow relative to a horizontal plane is characterized by angle Θ. The pressure variation in the "eye", region IV, is entirely axial and is denoted $p_o^*(z^*)$; for the idealization of a completely dry, cloud-free, fully developed eye, this variation is determined by dry-adiabatic compression of tropopause-level air.
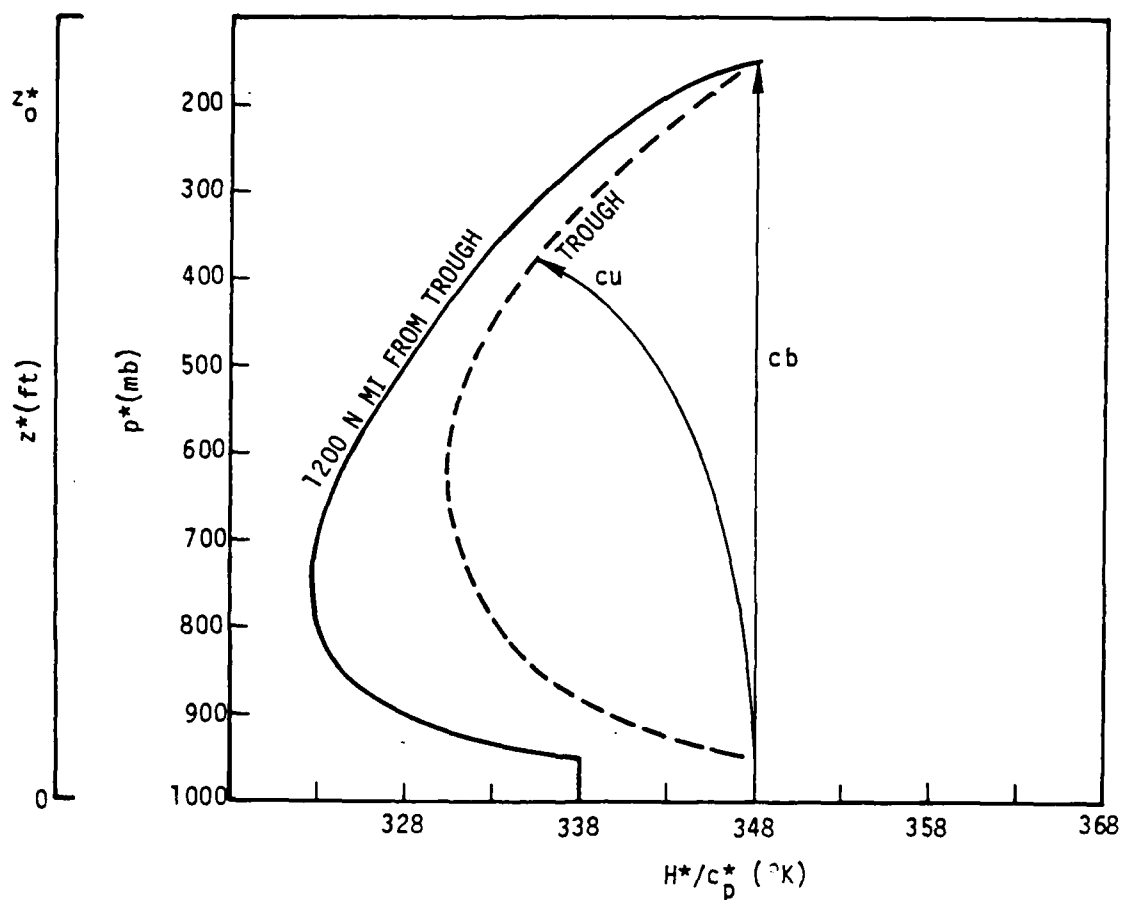
Figure 8. A typical profile of the total static temperature (H^*/c_p^*) over the ocean in the equatorial trough and in the subtropics (1200n mi from the equatorial trough), as a function of pressure p* and altitude above sea-level z*. The loci of thermodynamic states for a typical cumulus (cu) and also for the undiluted core of a cumulonimbus (cb) are also noted; the locus (cb) is that appropriate for moist-adiabatic ascent of sea-level air. The dashed curve is the typical ambient referred to in the text: a well-mixed lowest layer, a midtropospheric minimum, and a recovery of sea-level value at altitude z_0^* (defined here to be the tropopause, and typically about 50,000 ft). The quantity H* is the sum of the static enthalpy, the latent heat equivalent of the water vapor present, and the gravitational potential energy; the quantity c_p^* is the specific heat at constant pressure, which may be taken to be effectively that of dry air.

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